[20 points=5+5+10] **Problem 1.**

a) Find the Fourier series of the 2π-periodic function $\phi(x) = \sin x \cos x$ **without** doing the integral computations of the coefficients a_k and b_k – Justify your answer.

b) Suppose that f is a piecewise continuous 2π-periodic function. Let $g(x) = \int_0^x f(t)dt$. Give a necessary and sufficient condition on f so that g is 2π-periodic.

c) Knowing that the Fourier series of the odd function $sign(x) := \begin{cases} -1 & \text{for } -\pi \leq x < 0 \\ 0 & x = 0 \\ 1 & \text{otherwise} \end{cases}$

is given by $\frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{2k+1}$, find (after justifications) the Fourier series of the 2π-periodic function defined by $g(x) = |x|$ for $x \in [-\pi, \pi]$.

a) $\phi(x) = \sin x \cos x = \frac{1}{2} \sin 2x$

~~So~~ The Fourier series of ϕ is $\frac{1}{2}\sin 2x$

In other words: The FS is

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

where

$$0 = a_0 = a_1 = \dots .$$

& $$b_0 = b_1 = b_3 = b_4 = \dots = b_k = \dots = 0$$

$$b_2 = \frac{1}{2}.$$

$$b_k = 0 \quad \forall \quad k \neq 2 \quad \& \quad b_2 = \frac{1}{2}.$$

b) A necessary & sufficient condition on f would be that $\int_{-\pi}^{\pi} f(x) = 0$

This would lead to the periodicity of $g(x) = \int_0^x f(t)\,dt$.

c) $g(x) = |x|$

is the primitive of

$$\operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

~~g(x)~~ Notice that

$$\int_0^x \operatorname{sign}(t)\,dt = \int_0^x dt = x \quad \text{if } x > 0.$$

$$\int_0^x \operatorname{sign}(t)\,dt = 0 \quad \text{if } x = 0.$$

$$\int_0^x \operatorname{sign}(t)\,dt = \int_0^x -1\,dt \quad \text{if } x < 0$$

$$= -x \quad \text{if } x < 0.$$

Moreover $$\int_{-\pi}^{\pi} \operatorname{sign}(x)\,dx = 0$$

So we can integrate term by term to get

$$FS(|x|) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \int_0^x (\sin(2k+1)\pi t)\,dt$$

$$= \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{-1}{(2k+1)^2} (\cos(2k+1)\pi x + 1)$$

[25 points=20+ 5] **Problem 2.** Let $\tilde{f}$ be the 2π-periodic extension of the function f defined, over $[-\pi, \pi]$, by $f(x) = e^x$.

a) Compute the Fourier series of f.

b) What is the value of the Fourier series of f at the point $x = -\pi$? Is it $e^{-\pi}$?

a) $f(x) = e^x$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x\, dx = \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$$

for $k \geq 1$,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos kx\, dx$$

$$(*)\ \boxed{\int \underbrace{e^x}_{u} \underbrace{\cos kx\, dx}_{dv}} \overset{\text{by parts}}{=} \frac{1}{k} e^x \sin kx - \frac{1}{k}\int \underbrace{\sin kx}_{dv}\, \underbrace{e^x}_{u}\, dx$$

$$= \frac{e^x \sin kx}{k} - \frac{1}{k}\left[\frac{-e^x}{k}\cos kx + \int \frac{e^x}{k}\cos kx\, dx\right]$$

$$= \frac{e^x \sin kx}{k} - \boxed{\frac{1}{k^2}\int e^x \cos kx\, dx} + \frac{1}{k^2} e^x \cos kx$$

$$\int_{-\pi}^{\pi} e^x \cos kx\, dx \Big) \underbrace{\left(1 + \frac{1}{k^2}\right)}_{\frac{k^2+1}{k^2}} = \underbrace{\frac{e^x \sin kx}{k}\Big|_{-\pi}^{\pi}}_{0} + \frac{1}{k} e^x\left[\sin kx + \frac{1}{k}\cos kx\right]$$

$$+ \underbrace{\frac{1}{k} e^x \sin kx\Big|_{-\pi}^{\pi}}_{0} + \frac{1}{k^2} e^x \cos kx$$

$$\frac{1}{k^2}\left[e^{\pi}\cos k\pi - e^{-\pi}\cos k\pi\right] = \frac{(-1)^k}{k^2}\left[e^{\pi} - e^{-\pi}\right]$$

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos kx \, dx = \frac{(-1)^k}{(k^2+1)\pi}\left[e^{\pi}-e^{-\pi}\right].$$

from before (*) we see

$$-\frac{1}{k}\int_{-\pi}^{\pi} \sin kx \; e^x \, dx = \pi a_k - \cancelto{0}{\frac{1}{k} e^x \sin kx \Big|_{-\pi}^{\pi}}$$

so $\cancel{a_k}$

$$-\frac{1}{k}\cdot \cancel{\pi}\, b_k = \cancel{\pi}\, a_k .$$

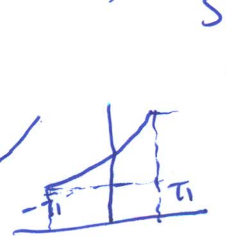
i.e $\boxed{b_k = -k a_k}$

$$b_k = -\frac{(-1)^k k}{(k^2+1)\pi}\left[e^{\pi}-e^{-\pi}\right].$$

$$FS(|x|) = \left[\frac{(e^{\pi}-e^{-\pi})}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2+1}\cos kx - \frac{(-1)^k k \sin kx}{(k^2+1)}\right]$$

$$+\frac{1}{2}\cdot\frac{1}{\pi}\left(e^{\pi}-e^{-\pi}\right).$$

b) From the convergence Theorem of Fourier Series, The value at $-\pi$ (which is a Discontinuity point) is $\frac{1}{2}\left[f(-\pi^-)+f(-\pi^+)\right]$



$$= \frac{1}{2}\left[e^{+\pi}+e^{-\pi}\right] \;\; " \; f(\pi) \,!$$

[30 points=10+10+5+5] Problem 3. Consider the following sequences of functions over the real line $\mathbb{R}$:

$$f_n(x) = xe^{-nx^2} \quad \text{and} \quad g_n(x) = nxe^{-nx^2}, \quad n = 1, 2, \cdots$$

1. Determine the pointwise limits of the the sequences of functions $\{f_n(x)\}_n$ and $\{g_n(x)\}_n$ for all $x \in \mathbb{R}$?
2. Compute $\max_{x\in\mathbb{R}} |f_n(x)|$ and $\max_{x\in\mathbb{R}} |g_n(x)|$.
3. Study the uniform convergence of $\{f_n(x)\}_n$ over $\mathbb{R}$. Justify your answer.
4. Study the uniform convergence of $\{g_n(x)\}_n$ over $\mathbb{R}$. Justify your answer.

1) Fixing a real number $x \in \mathbb{R}$:

if $x = 0$, $f_n(0) = 0 \xrightarrow[n\to\infty]{} 0$.

if $x \neq 0$, then $e^{-nx^2} \xrightarrow[n\to\infty]{} 0$ (since $x^2 > 0$).

Thus $f_n(x) \xrightarrow[n\to\infty]{} 0$ when $x \neq 0$

& also when $x = 0$.

i.e $f_n \longrightarrow 0$ pointwise.

Also when $x \neq 0$, $nx\, e^{-nx^2} \xrightarrow[n\to\infty]{} 0$. This implies $g_n(x) \longrightarrow 0$ pointwise.

2. $|f_n(x)| = |x e^{-nx^2}|$, note that $x \longmapsto x e^{-nx^2}$ is an ODD function.

So It is enough to compute $\max_{x \geq 0} x e^{-nx^2}$.

$f_n'(x) = 1.\, e^{-nx^2} - 2nx^2 e^{-nx^2} = e^{-nx^2}[1 - 2nx^2]$

$f_n'(x) = 0 \iff x = \mp \frac{1}{\sqrt{2n}}$.

x	$-\infty$		$-1/\sqrt{2n}$		0		$1/\sqrt{2n}$		$+\infty$
$f_n'(x)$		$-$	0	$+$		$+$	0	$-$	

$\max_{x \geq 0} f_n(x)$

Therefore $\max_{x\in\mathbb{R}} |f_n(x)| = f_n\left(\frac{1}{\sqrt{2n}}\right)$

$$= \frac{1}{\sqrt{2n}} e^{-n\cdot\frac{1}{2n}}$$

$$\boxed{= \frac{1}{\sqrt{2n}} e^{-1}}$$

For g_n: $g_n(x)$ is also ODD.

Let's then consider $\max_{x\geq 0} g_n(x)$

instead $\max_{x\in\mathbb{R}} |g_n(x)|$.

$$g_n'(x) = \left(nx\, e^{-nx^2}\right)' = n\, e^{-nx^2} + nx.(-2nx)e^{-nx^2}$$

$$= e^{-nx^2}\left[n - 2n^2x^2\right].$$

$$g_n'(x) = 0 \iff x = \mp\frac{1}{\sqrt{2n}}.$$

x	$-\infty$		$-1/\sqrt{2n}$		0		$1/\sqrt{2n}$		$+\infty$
$g_n'(x)$		$-$	0	$+$		$+$	0	$-$	

$$\max_{x\in\mathbb{R}} |g_n(x)| = g_n\left(\frac{1}{\sqrt{2n}}\right) = n\cdot\frac{1}{\sqrt{2n}}\, e^{-n\cdot\frac{1}{2n}}$$

$$= \frac{\sqrt{n}}{\sqrt{2}} e^{-1}.$$

3) We notice that:

$$\| f_n - 0 \|_\infty = \max_{x \in \mathbb{R}} | f_n(x) - 0 |$$

$$= \frac{e^{-1}}{\sqrt{2n}} \xrightarrow[n \to \infty]{} 0$$

Thus $f_n \longrightarrow 0$ Uniformly.

4) $$\max_{x \in \mathbb{R}} | g_n(x) - 0 |$$

$$= \frac{e^{-1}}{\sqrt{2}} \cdot \sqrt{n} \xrightarrow[n \to \infty]{} +\infty$$

So $\{g_n\}$ Does NOT converge to 0 Uniformly

(only Pointwise)

[25 points=10+5+5+5] Problem 4. Consider the following PDE whose unknown is $u(t,x)$

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad u(0,x) = f(x), u_x(0,x) = g(x) \tag{1}$$

where $t \geq 0$ and $x \in \mathbb{R}$ and f, g are two given functions of class C^2 over $\mathbb{R}$. Let

$$r = x + t \text{ and } s = x - t \text{ and } u(t,x) = v(r,s) = v(x+t, x-t).$$

1. Use the cain rule to find $\frac{\partial^2 u}{\partial t^2}(t,x)$ and $\frac{\partial^2 u}{\partial x^2}$ in terms of the partial derivative v_s, v_r, v_{rr}, v_{rs} and v_{ss}.

2. Show that the PDE (1) is equivalent to the PDE

$$\frac{\partial^2 v}{\partial r \partial s} = 0. \tag{2}$$

3. Write down the general solution of (2) (**Help:** $\frac{\partial^2 v}{\partial r \partial s} = 0$ means that $\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial s}\right) = 0$).

4. Use the initial data on $u(t,x)$ and part 3. to find the form of the solution of (1) interns of $f(x)$ and $g(x)$.

1) $$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial t} + \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial t}$$

$$= v_r \cdot 1 + v_s \cdot (-1) = v_r - v_s.$$

$$u_{tt} = v_{rr} - v_{rs} - v_{rs} + v_{ss}$$

$$= v_{rr} - 2 v_{rs} + v_{ss}.$$

$$u_x = v_r \cdot \frac{\partial r}{\partial x} + v_s \cdot \frac{\partial s}{\partial x} = v_r \cdot 1 + v_s.$$

$$u_{xx} = v_{rr} + v_{rs} + v_{sr} + v_{ss}$$

$$= v_{rr} + 2 v_{rs} + v_{ss}.$$

2) $$u_{tt} = u_{xx} \Rightarrow -4 v_{rs} = 0$$

$$\Rightarrow \boxed{v_{rs} = 0}$$

3) (2) can be written as

$$\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial s}\right) = 0$$

So $\frac{\partial v}{\partial s} = \phi(s)$ for some function ϕ.

Integrating in s, leads to

$$v(r,s) = \int \phi(s)\,ds + \psi(r)$$

$$= q(s) + \psi(r)$$

for some functions, q & ψ

where $q'(s) = \phi(s)$.

$$v(r,s) = q(s) + \psi(r)$$

$$\Rightarrow \boxed{u(t,x) = q(x-t) + \psi(x+t)}$$

4) $u(0,x) = f(x) \Rightarrow f(x) = q(x) + \psi(x) \quad \ldots (\alpha)$

$u_x(0,x) = g(x) \Longrightarrow g(x) = q'(x) + \psi'(x) \ldots (*)$

Note: $\frac{\partial}{\partial x}(q(x-t))\Big|_{t=0} = q'(x-t)\cdot 1\Big|_{t=0}$
$= q'(x)$.

$(*) \Rightarrow \int_0^x g(t)\,dt = q(x) + \psi(x) - q(0) - \psi(0) \quad \ldots (\beta)$

$(\alpha) \Rightarrow \boxed{\int_0^x g(t)\,dt + q(0) + \psi(0) = f(x)}$